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Perimeter Defense Game with Nonzero Capture Radius in a Circular Target

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1. Problem statement

This paper formulates the target guarding problem wherein the Defender (D) constrained to move along the circular target perimeter and the Attacker (A) moves in the plane with simple motion. The Defender can make interception with r capture radius. On figure 1 can be seen the illustration and the rules of the game. The goal of the Attacker is entering the target without interception, and the goal of the Defender is preventing the penetration. Selected assumptions are made on the

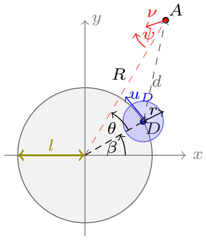

Figure 1. The illustration of the perimeter defense game with nonzero capture radius

problem statement:

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Assumption 1. *The target is a circle with $l = 1$ radius.*

Assumption 2. *The player's speeds are such that $0 < \nu \leq u_D = 1$, where ν is the speed of Attacker and u_D is the speed of Defender.*

Assumption 3. *The Defender make interception with r capture radius. The $\mathcal{C}$ Capture Circle is defined as the set of the states of satisfying*

$$\mathcal{C} = \{(R, \theta) \,|\, r^2 \leq R^2 + 1 - 2R\cos\theta\} \tag{1}$$

The kinematics can be written as

$$f(\mathbf{x}, u, t) = \dot{\mathbf{x}} = \begin{bmatrix} \dot{R} \\ \dot{\theta} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -\nu\cos\psi \\ \nu\frac{1}{R}\sin\psi - u_D \\ u_D \end{bmatrix} \tag{2}$$

The Defender control lies in the range $u_D \in [-1,\, 1]$, and the Attacker control lies in the range $\psi \in [-\pi\,,\pi]$.

The Defender wins scenario, when D is able to make interception before A can reach the target, the agents play zero sum game over the cost-functional

$$\Phi_d(\mathbf{x}_f, t_f) = -R_f = J_d, \tag{3}$$

where the subscript f denotes the termination. The Defender is the minimizing player and the Attacker is the maximizing player. The Value of the game if it exists, is the saddle-point equilibrium of the cost-functional over state-feedback strategies

$$V_d = \min_{u_D(\cdot)} \max_{\psi(\cdot)} J_d = \max_{\psi(\cdot)} \min_{u_D(\cdot)} J_d. \tag{4}$$

The terminal constraint is

$$\phi_d(\mathbf{x}_f, t_f) = \sqrt{R^2 + 1 - 2R\cos\theta} - r = d - r = 0. \tag{5}$$

The final time t_f is the first time for which $d = r$. Thus, the Terminal Surface is defined as the set of states of satisfying (5)

$$\mathcal{J}_d = \{\mathbf{x} \,|\, R > 1 \quad \text{and} \quad d = r\}. \tag{6}$$

The Attacker wins scenario, when A is able to drive $R \longrightarrow 1$ while avoiding $d \leq r$, the agents play zero sum game over the cost-functional

$$J_a = \Phi_a(\mathbf{x}_f, t_f) = d - r. \tag{7}$$

The Defender is the minimizing player and the Attacker is the maximizing player. The Value of the game if it exists, is the saddle-point equilibrium of the cost-functional over state-feedback strategies

$$V_a = \min_{u_D(\cdot)} \max_{\psi(\cdot)} J_a = \max_{\psi(\cdot)} \min_{u_D(\cdot)} J_a. \tag{8}$$

Termination occurs when the Attacker reaches the target circle, therefore the termination constraint

$$\phi(\mathbf{x}_f, t_f) = R_f - 1 = 0. \tag{9}$$

The final time t_f is the first time for which $R(t) = 1$. Thus, the Terminal Surface is defined as the set of states of satisfying (9)

$$\mathcal{J}_a = \{\mathbf{x} \,|\, R = 1 \quad \text{and} \quad d \geq r\}. \tag{10}$$

2. Methods and results

The steps of the analytic solution of the Defender wins scenerio follow the steps of the zero capture radius case [3]. The analysis is carried out according to a classical differential game approach [2] [1]. The solution of Attacker wins scenerio based upon showing satisfaction of the sufficient condition for equilibrium via substitution of the proposed equilibrium strategies of the Defender wins scenario and Value function into the Hamilton-Jacobi-Isaacs equation [2].

In a Defender wins and Attacker wins scenario the agents have the same equilibrium strategies: the Attacker moves the tangent of the ν radius circle and the defender moves along the perimeter of the target towards the Attacker. The equilibrium flow field shows the winning regions and the trajectories of the states in the two game: Defender and Attacker wins scenario. Figure 2 shows the full equilibrium flow field in case $\nu = 0.8$, $r = 0.4$. The Attacker winning region and the trajectories denoted by red, the Defender winning region and the trajectories denoted by orange, the trajectory of limiting case denoted by black and the terminal surface of Defender wins scenario denoted bay olive.

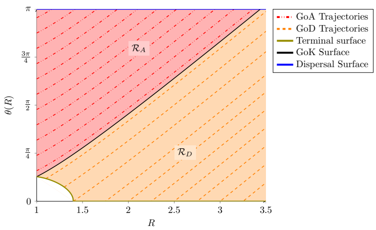

Figure 2. Full equilibrium flow field with $\nu = 0.8$ and $r = 0.4$

References

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[3] A. Von Moll, M. Pachter, D. Shishika, Z. Fuchs: *Circular target defense differential games*, IEEE Transactions on Automatic Control 68.7 (2022), pp. 4065–4078.